

# DISTANCE SETS OF WELL-DISTRIBUTED PLANAR SETS FOR POLYGONAL NORMS

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ABSTRACT. Let  $X$  be a 2-dimensional normed space, and let  $BX$  be the unit ball in  $X$ . We discuss the question of how large the set of extremal points of  $BX$  may be if  $X$  contains a well-distributed set whose distance set  $\Delta$  satisfies the estimate  $|\Delta \cap [0, N]| \leq CN^{3/2-\epsilon}$ . We also give a necessary and sufficient condition for the existence of a well-distributed set with  $|\Delta \cap [0, N]| \leq CN$ .

## §0. INTRODUCTION

The classical Erdős Distance Problem asks for the smallest possible cardinality of

$$\Delta(A) = \Delta_{l_2^2}(A) = \left\{ \|a - a'\|_{l_2^2} : a, a' \in A \right\}$$

if  $A \subset \mathbb{R}^2$  has cardinality  $N < \infty$  and

$$\|x\|_{l_2^2} = \sqrt{x_1^2 + x_2^2}$$

is the Euclidean distance between the points  $a$  and  $a'$ . Erdős conjectured that  $|\Delta(A)| \gg N/\sqrt{\log N}$  for  $N \geq 2$ . (We write  $U \ll V$ , or  $V \gg U$ , if the functions  $U, V$  satisfy the inequality  $|U| \leq CV$ , where  $C$  is a constant which may depend on some specified parameters). The best known result to date in two dimensions is due to Katz and Tardos who prove in [KT04] that  $|\Delta(A)| \gg N^{.864}$  improving an earlier breakthrough by Solymosi and Tóth [ST01].

More generally, one can examine an arbitrary two-dimensional space  $X$  with the unit ball

$$BX = \{x \in \mathbb{R}^2 : \|x\|_X \leq 1\}$$

and define the distance set

$$\Delta_X(A) = \{\|a - a'\|_X : a, a' \in A\}.$$

For example, let

$$\|x\|_{l_\infty^2} = \max(|x_1|, |x_2|)$$

then for  $N \geq 1$ ,  $A = \{m \in \mathbb{Z}^2 : 0 \leq m_1 \leq N^{1/2}, 0 \leq m_2 \leq N^{1/2}\}$  we have  $|A| \gg N$ ,  $|\Delta_{l_\infty^2}(A)| \ll N^{1/2}$ . This simple example shows that the Erdős Distance Conjecture can not be directly extended for arbitrary two-dimensional spaces. Erdős [E46] (see also [I01]) proved the estimate  $|\Delta_X(A)| \gg N^{1/2}$  for any space  $X$ .

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Also, for a positive integer  $N$  we denote

$$\Delta_{X,N}(A) = \{\|a - a'\|_X \leq N : a, a' \in A\}.$$

We say that a set  $S \subset X$  is well-distributed if there is a constant  $K$  such that every closed ball of radius  $K$  in  $X$  contains a point from  $S$ . In other words, for every point  $x \in X$  there is a point  $y \in S$  such that  $\|x - y\|_X \leq K$ . Sometimes it is said that  $S$  is a  $K$ -net for  $X$ . Clearly, for any well-distributed set  $S$  and  $N \geq 2K$  we have

$$(1) \quad |\{x \in S : \|x\|_X \leq N/2\}| \gg N^2$$

where the constant in  $\gg$  depends only on  $K$ . Therefore, for any well-distributed set  $S \in l_2^2$  we have, by [T02],

$$|\Delta_{l_2^2,N}(S)| \gg N^{1.728},$$

and the Erdős Distance Conjecture implies for large  $N$

$$|\Delta_{l_2^2,N}(S)| \gg N^2 / \sqrt{\log N}.$$

On the other hand, for a well-distributed set  $S = \mathbb{Z}^2 \subset l_\infty^2$  we have

$$|\Delta_{l_\infty^2,N}(S)| = 2N + 1.$$

Iosevich and the second author [IL03] have recently established that a slow growth of  $|\Delta_{X,N}(S)|$  for a well-distributed set  $S \subset X$  is possible only in the case if  $BX$  is a polygon with finitely or infinitely many sides. Let us discuss possible definitions of polygons with infinitely many sides. For a convex set  $A \subset X$  by  $Ext(A)$  we denote the set of extremal points of  $A$ . Namely,  $x \in Ext(A)$  if and only if  $x \in A$  and for any segment  $[y, z]$  the conditions  $x \in [y, z] \subset A$  imply  $x = y$  or  $x = z$ . Clearly,  $Ext(BX)$  is a closed subset of the unit circle

$$\partial BX = \{x \in X : \|x\|_X = 1\}.$$

Also, it is easy to see that  $Ext(BX)$  is finite if and only if  $BX$  is a polygon with finitely many sides, and it is natural to consider  $BX$  as a polygon with infinitely many sides if  $Ext(BX)$  is small. There are different ways to define smallness of  $Ext(BX)$  and, thus, polygons with infinitely many sides:

- 1) in category:  $Ext(BX)$  is nowhere dense in  $\partial BX$ ;
- 2) in measure:  $Ext(BX)$  has a zero linear measure (or a small Hausdorff dimension);
- 3) in cardinality:  $Ext(BX)$  is at most countable.

Clearly, 3) implies 2) and 2) implies 1).

It has been proved in [IL03] that the condition

$$(0.1) \quad \varliminf_{N \rightarrow \infty} |\Delta_{X,N}(S)| N^{-3/2} = 0$$

for a well-distributed set  $S$  implies that  $BX$  is a polygon in a category sense. Following [IL03], we prove that, moreover,  $BX$  is a polygon in a measure sense.

**Theorem 1.** *Let  $S$  be a well-distributed set.*

- (i) *Assume that (0.1) holds. Then the one-dimensional Hausdorff measure of  $\text{Ext}(BX)$  is 0;*
- (ii) *If moreover*

$$(0.2) \quad |\Delta_{X,N}(S)| = O(N^{1+\alpha})$$

*for some  $\alpha \in (0, 1/2)$  then the Hausdorff dimension of  $\text{Ext}(BX)$  is at most  $2\alpha$ .*

If  $|\Delta_{X,N}(S)|$  has an extremally slow rate of growth for some well-distributed set  $S$ , namely,

$$(0.3) \quad |\Delta_{X,N}(S)| = O(N)$$

then, as it has been proved in [IL03],  $BX$  is a polygon with finitely many sides. However, if we weaken (0.3) we cannot claim that  $BX$  is a polygon in a cardinality sense.

**Theorem 2.** *Let  $\psi(u)$  be a function  $(0, \infty) \rightarrow (0, \infty)$  such that  $\lim_{u \rightarrow \infty} \psi(u) = \infty$ . Then there exists a 2-dimensional space  $X$  and a well-distributed set  $S \subset X$  such that*

$$(0.4) \quad |\Delta_{X,N}(S)| = o(N\psi(N)) \quad (N \rightarrow \infty)$$

*but  $\text{Ext}(BX)$  is a perfect set (and therefore is uncountable).*

Also, we find a necessary and sufficient condition for a space  $X$  to make (0.3) possible for some well-distributed set  $S \subset X$ . Take two non-collinear vectors  $e_1, e_2$  in  $X$ . They determine coordinates for any  $x \in X$ , namely,  $x = x_1 e_1 + x_2 e_2$ . Then, for any non-degenerate segment  $I \subset X$ , we can define its slope  $Sl(I)$ : if the line containing  $I$  is given by an equation  $u_1 x_1 + u_2 x_2 + u_0 = 0$ , then we set  $Sl(I) = -u_1/u_2$ . We write  $Sl(I) = \infty$  if  $u_2 = 0$ ; it will be convenient for us to consider  $\infty$  as an algebraic number.

**Theorem 3.** *The following conditions on  $X$  are equivalent:*

- (i)  *$BX$  is a polygon with finitely many sides, and there is a coordinate system in  $X$  such that the slopes of all sides of  $BX$  are algebraic;*
- (ii) *there is a well-distributed set  $S \subset X$  such that (0.3) holds.*

**Corollary 1.** *If a norm  $\|\cdot\|_X$  on  $\mathbb{R}^2$  is so that  $BX$  is a polygon with finitely many sides and all angles between its sides are rational multiples of  $\pi$  then there is a well-distributed set  $S \subset X$  such that (0.3) holds.*

**Corollary 2.** *If a norm  $\|\cdot\|_X$  on  $\mathbb{R}^2$  is defined by a regular polygon  $BX$  then there is a well-distributed set  $S \subset X$  such that (0.3) holds.*

The Falconer conjecture (for the plane) says that if the Hausdorff dimension of a compact  $A \subset \mathbb{R}^2$  is greater than 1 then  $\Delta(A)$  has positive Lebesgue measure. The best known result is due to Wolff who proved in [W99] that the distance set has positive Lebesgue measure if the Hausdorff dimension of  $A$  is greater than  $4/3$ . One can ask a similar question for an arbitrary two-dimensional normed space  $X$ . It turns out that this question is related to distance sets for well-distributed and separated sets. By Theorem 4 from [IL04], Theorem 3 and Proposition 1 we get the following.

**Corollary 3.** *If a norm  $\|\cdot\|_X$  on  $\mathbb{R}^2$  is defined by a polygon  $BX$  with finitely many sides all of which have algebraic slopes then there is a compact  $A \subset X$  such that the Hausdorff dimension of  $A$  is 2 and Lebesgue measure of  $\Delta_X(A)$  is 0.*

It would be interesting to know if the result is true without supposition on the slopes of the sides.

Recall that, by [IL03], it is enough to prove the implication (ii)  $\rightarrow$  (i) in Theorem 3 assuming that  $BX$  is a polygon. In that case we prove a stronger result.

**Theorem 4.** *Let  $BX$  be a polygon with finitely many sides which does not satisfy the condition (i) of Theorem 3. Then for any well-distributed set  $S$  we have*

$$(0.5) \quad |\Delta_{X,N}(S)| \gg N \log N / \log \log N \quad (N \geq 3).$$

Comparison of Theorem 4 with Theorem 2 shows that the growth of  $|\Delta_{X,N}(S)|$  for well-distributed sets and  $N \rightarrow \infty$  does not distinguish the spaces  $X$  with small and big cardinality of  $\text{Ext}(BX)$ .

### §1. PROOF OF THEOREMS 1 AND 2

*Proof of (i).* Without loss of generality we may assume that  $BX \subset Bl_2^2$  and the set  $S$  is well-distributed in  $X$  with the constant  $K = 1/2$ . Also, choose  $\delta > 0$  so that

$$(1.1) \quad \delta Bl_2^2 \subset BX.$$

By (0.1), for any  $\varepsilon > 0$  there are arbitrary large  $N_0$  such that

$$|\Delta_{X,N_0}(S)| \leq \varepsilon N_0^{3/2}.$$

If  $N_0 \geq 8$  then the number of integers  $j \geq 0$  with  $N_0/2 + 4j \leq N_0 - 2$  is

$$\geq (N_0/2 - 2)/4 \geq N_0/8.$$

Thus, there is at least one  $j$  such that  $N = N_0/2 + 4j$  satisfies the condition

$$(1.2) \quad |(\Delta_X(S)) \cap (N - 2, N + 2)| \leq 8\varepsilon N_0^{3/2}/N_0 \leq 12\varepsilon N^{1/2}.$$

So, (1.2) holds for arbitrary large  $N$ .

We take any  $N$  satisfying (1.2) and an arbitrary  $P \in S$ . Let  $Q$  be the closest point to  $P$  in the space  $X$  (observe that it exists since  $S$  is closed due to (0.1)). Then, by well-distribution of  $S$  (recall that  $K = 1/2$ ) we have

$$(1.3) \quad \|P - Q\|_X \leq 1.$$

Without loss of generality,  $P = 0$ . Denote  $M = [2N\delta]$  and consider the rays

$$L_j = \{(r, \theta) : \theta = \theta_j = 2\pi j/M\},$$

where  $(r, \theta)$  are the polar coordinates in  $l_2^2$ . Consider a point  $R_j$ ,  $1 \leq j \leq M$ , with the polar coordinates  $(r_j, (\theta_{j-1} + \theta_j)/2)$  such that  $\|R_j\|_X = N$ . By (1.1) we have

$$r_j \geq \delta N.$$

Therefore, the Euclidean distance from  $R_j$  to the rays  $L_{j-1}$  and  $L_j$  is

$$(1.4) \quad r_j \sin(\pi/M) \geq N\delta \sin(\pi/(2N\delta)) > 1.$$

provided that  $N$  is large enough. Therefore, the distance from  $R_j$  to these rays in  $X$  is also greater than 1. Also, the distance from  $R_j$  to the circles

$$\Gamma_1 = \{R : \|R\|_X = N - 1\}, \quad \Gamma_2 = \{R : \|R\|_X = N + 1\}$$

in  $X$  is equal to 1. Thus, the  $X$ -disc of radius  $1/2$  with the center at  $R_j$  is contained in the open region  $U_j$  bounded by  $L_{j-1}$ ,  $L_j$ ,  $\Gamma_1$ , and  $\Gamma_2$ . By the supposition on  $S$  there is a point  $P_j \in U_j \cap S$ .

Observe that for any  $j$  we have

$$N - 1 < \|P - P_j\|_X < N + 1, \quad N - 2 < \|Q - P_j\|_X < N + 2.$$

Let  $U = \{(\|P - P_j\|_X, \|Q - P_j\|_X)\}$ . By (1.2),

$$(1.5) \quad |U| \leq 144\varepsilon^2 N.$$

For any  $(n_1, n_2) \in U$  we denote

$$J_{n_1, n_2} = \{j : \|P - P_j\|_X = n_1, \quad \|Q - P_j\|_X = n_2\}.$$

By [IL03, Lemma 1.4, (i)], if  $j_1, j_2, j_3 \in J_{n_1, n_2}$  then one of the points  $P_{j_1}, P_{j_2}, P_{j_3}$  must lie on the segment connecting two other points and contained in the circle  $\{R : \|P - R\|_X = n_1\}$ . This implies that for all  $j \in J_{n_1, n_2}$  but at most two indices the intersection of  $\partial BX$  with the sector  $S_j$  bounded by  $L_{j-1}$  and  $L_j$  is inside some line segment contained in  $\partial BX$ . Therefore, by (1.5), the number of sectors  $S_j$  containing an extremal point of  $BX$  is at most  $288\varepsilon^2 N$ . For  $R \in \partial BX$  with the polar coordinates  $(r, \theta)$  denote  $\Theta(R) = \theta$ . Define the measure on  $\partial BX$  in such a way that for any Borel set  $V \subset \partial BX$  the measure  $\mu_P(V)$  is defined as the Lebesgue measure of  $\Theta(V)$ . In particular,

$$\mu_P(\partial BX \cap S_j) = \frac{2\pi}{M}.$$

Clearly,  $\mu_P$  is equivalent to the standard Lebesgue measure on  $\partial BX$ . We have proved that

$$\mu_P(\text{Ext}(BX)) \leq 288\varepsilon^2 N \frac{2\pi}{M}.$$

But  $1/M \leq 1/(N\delta)$ . Hence,

$$\mu_P(\text{Ext}(BX)) \leq 2\pi \times 288\varepsilon^2 / \delta.$$

As  $\varepsilon$  can be chosen arbitrarily small, we get  $\mu_P(\text{Ext}(BX)) = 0$ , and this completes the proof of (i).

Proof of (ii) follows the same scheme. Inequality (1.2) should be replaced by

$$|\Delta_X(S) \cap (N - 2, N + 2)| \leq \Delta N^\alpha,$$

where  $\Delta$  may depend only on  $X$ ,  $S$ , and  $\alpha$ . We define the distance  $d_p$  on  $\partial BX$  as the distance between the polar coordinates. This metric is equivalent to the  $X$ -metric. The set  $Ext(BX)$  can be covered by at most  $2\Delta^2 N^{2\alpha}$  arcs  $\partial BX \cap S_j$  each of them has the  $d_p$ -diameter at most  $2\pi/(N\delta)$ . This implies the required estimate for the Hausdorff dimension of  $Ext(BX)$ .

*Proof of Theorem 2.* We select an increasing sequence  $\{N_j\}$  of positive integers such that

$$(1.6) \quad \psi(N) \geq 5^j \quad (N \geq N_j).$$

By  $\Lambda_j$  we denote the set of numbers  $a/q$  with  $a \in \mathbb{Z}$ ,  $q \in \mathbb{N}$ ,  $q \leq N_j$ . We will construct a ball  $BX$  on the Euclidean plane. Moreover, it will be symmetric with respect to the lines  $x_1 = x_2$  and  $x_1 = -x_2$ , and thus it suffices to construct  $BX$  in the quadrant  $Q = \{x : x_2 \geq |x_1|\}$ .

Let  $D_0$  be the square

$$D_0 = \{x : 0 \leq x_2 + x_1 \leq 1, 0 \leq x_2 - x_1 \leq 1\}.$$

We will construct a decreasing sequence of polygons  $D_j$ ; each one will be defined as a result of cutting some angles from the previous one. The sides  $V_1, V_2$  of  $D_0$  with an endpoint at the origin will not be changed. The intersection of the sequence  $D_j$  will define the part of our  $BX$  in  $Q$ . In particular, the points  $(\pm 1/2, 1/2)$  will be vertices of all polygons  $D_j$ . Therefore, these points as well as the symmetrical points  $(\pm 1/2, -1/2)$  will be in  $\partial BX$ .

First, we construct  $D_1$  as a result of cutting  $D_0$  by a line  $x_2 = u$  for some  $u \in (1/2, 1)$ . We choose  $u$  such that for intersection points  $x^1$  and  $x^2$  of this line with the boundary of  $D_0$  the ratios  $x_1^j/x_2^j$  ( $j = 1, 2$ ) differs from all numbers  $\lambda \in \Lambda_1$ . Moreover, we take neighborhoods  $U_j$  of the points  $x^j$  ( $j = 1, 2$ ) such that

$$\forall y \in U_j \quad y_2/y_1 \notin \Lambda_1 \quad (j = 1, 2).$$

In the sequel we shall make other cuts only inside the sets  $U_1$  and  $U_2$ . This means that all points  $x$  on the boundary of  $D_1$  with  $x_1/x_2 \in \Lambda_1$  not belonging to the sides  $V_1, V_2$  as well as their neighborhoods in the boundary of  $D_1$  will remain in all polygons  $D_2, D_3, \dots$ , and eventually they will be interior points of some segments in the boundary of  $BX$  with a slope  $-1, 0$ , or  $1$ ,

On the second step, we construct  $D_2$  as a result of cutting  $D_1$  by lines with slopes  $-1/2$  and  $1/2$  such that for any new vertex  $x$  of a polygon  $D_2$  we have  $x_2/x_1 \notin \Lambda_2$ . Moreover, we take neighborhoods  $U(x)$  of all these points  $x$  (each is contained in  $U_1$  or in  $U_2$ ) such that

$$\forall y \in U(x) \quad y_2/y_1 \notin \Lambda_2.$$

Again, we shall make other cuts only inside the sets  $U(x)$ . This means that all points  $x$  on the boundary of  $D_2$  with  $x_1/x_2 \in \Lambda_2$  not belonging to the sides  $V_1, V_2$  as well as their neighborhoods in the boundary of  $D_2$  will remain in all polygons  $D_3, D_4, \dots$ , and eventually they will be interior points of some segments in the boundary of  $BX$  with a slope  $a/2$ ,  $a \in \mathbb{Z}$ ,  $|a| \leq 2$ .

Proceeding in the same way, we shall get a ball  $BX$  with the following property: if  $x \in \partial BX$  and  $x_1/x_2 \in \Lambda_{j+1}$  for some  $j$  then  $x$  is an interior point of some segment

contained in  $\partial BX$  with a slope  $a/2^j$ ,  $a \in \mathbb{Z}$ ,  $|a| \leq 2^j$ . This segment is a part of a line  $2^j x_2 - ax_1 = b(a, j)$  or a symmetrical line  $2^j x_2 - ax_1 = -b(a, j)$ . Also, by symmetry, if  $x \in \partial BX$  and  $x_2/x_1 \in \Lambda_{j+1}$  for some  $j$  then  $2^j x_1 - ax_2 = b(a, j)$  or  $2^j x_1 - ax_2 = -b(a, j)$ . In terms of the norm  $\|\cdot\|_X$  we conclude that if  $x \in X$  and  $x_1/x_2 \in \Lambda_{j+1}$  or  $x_2/x_1 \in \Lambda_{j+1}$  then  $\|x\|_X$  is equal to one of the numbers  $|2^j x_1 - ax_2|/|b(a, j)|$  or  $|2^j x_2 - ax_1|/|b(a, j)|$ ,  $a \in \mathbb{Z}$ ,  $|a| \leq 2^j$ . Also, observe that, by our construction,  $BX$  is contained in the square  $[-1, 1]^2$ . Therefore,

$$(1.7) \quad \|x\|_X \geq \max(|x_1|, |x_2|).$$

Now let us take the lattice  $S = \mathbb{Z}^2$  and estimate  $|\Delta_{X,N}(S)|$  for  $N_j < N \leq N_{j+1}$ . If  $x, y \in S$  and  $\|x - y\|_X \leq N$ , then we have  $\|x - y\|_X = |(z_1, z_2)|_X$  where  $z_1, z_2 \in \mathbb{Z}$  and, by (1.7),  $\max(|z_1|, |z_2|) \leq N$ . Hence,  $(z_1, z_2) = (0, 0)$ , or  $x_1/x_2 \in \Lambda_{j+1}$ , or  $x_2/x_1 \in \Lambda_{j+1}$ . Therefore,  $\|x - y\|_X = 0$  or  $\|x - y\|_X$  is equal to one of the numbers  $|2^j x_1 - ax_2|/|b(a, j)|$  or  $|2^j x_2 - ax_1|/|b(a, j)|$ ,  $a \in \mathbb{Z}$ ,  $|a| \leq 2^j$ . For every  $a$  we have

$$|2^j x_1 - ax_2| \leq 2^j |x_1| + |a| \times |x_2| \leq 2^{j+1} N.$$

Taking the sum over all  $a$  we get

$$(1.8) \quad |\Delta_{X,N}(S)| \leq (2^{j+1} + 1)2^{j+1}N + 1 \leq 2^{2j+3}N.$$

On the other hand, by (1.6),

$$(1.9) \quad \psi(N) \geq 5^j.$$

Comparing (1.8) and (1.9), we get (0.4) and thus complete the proof of the theorem.

## §2. PROOF OF THEOREM 3, PART I

In this section we prove that the condition (i) of Theorem 3 implies (ii).

Assume that  $\partial BX$  consists of a finite number of line segments with slopes  $\beta_1, \beta_2, \dots, \beta_r$ , all real and algebraic. Let  $\mathbb{F}_\mathbb{Q}[\beta_1, \dots, \beta_r]$  be the field extension of  $\mathbb{Q}$  generated by  $\beta_1, \dots, \beta_r$ , and let  $\alpha_0$  be its primitive element, i.e. an algebraic number such that  $\mathbb{F}_\mathbb{Q}[\beta_1, \dots, \beta_r] = \mathbb{F}_\mathbb{Q}[\alpha_0]$ . We may assume that  $\alpha_0$  is an algebraic integer: indeed, if  $\alpha_0$  is a root of  $P(x) = a_d x^d + \dots + a_0$ , then  $\alpha'_0 = a_d \alpha_0$  is a root of  $a_d^{d-1} P(x/a_d) = x^d + a_{d-1} x^{d-1} + a_{d-2} a_d x^{d-2} + \dots + a_0 a_d^{d-1}$ , hence an algebraic integer, and generates the same extension.

It suffices to prove that there is a well-distributed set  $S \subset \mathbb{R}^2$  such that

$$(2.1) \quad |\{x + \beta y : (x, y) \in S - S, |x| + |y| \leq R\}| \ll R,$$

for each  $\beta \in \mathbb{F}_\mathbb{Q}[\alpha]$ .

Since  $\mathbb{F}_\mathbb{Q}[\beta_1, \dots, \beta_r] \subset \mathbb{R}$ , we have  $\alpha_0 \in \mathbb{R}$ . Let  $\alpha_1, \dots, \alpha_{d-1}$  be the algebraic conjugates of  $\alpha_0$  in  $\mathbb{C}$  (of course they need not belong to  $\mathbb{F}_\mathbb{Q}[\alpha_0]$ ). Define for  $C > 0$

$$T(C) = \left\{ \sum_{j=0}^{d-1} a_j \alpha_0^j : a_j \in \mathbb{Z}, \left| \sum_{j=0}^{d-1} a_j \alpha_k^j \right| \leq C, k = 1, \dots, d-1 \right\},$$

and

$$S = T(C) \times T(C),$$

where  $C$  will be fixed later.

We first claim that  $T(C)$  is well distributed in  $\mathbb{R}$  (with the implicit constant dependent on  $C$ ), and that

$$(2.2) \quad |T(C) \cap [-R, R]| \ll R.$$

Indeed, let  $x = (x_0, x_1, \dots, x_{d-1})^T$  solve

$$\sum_{j=0}^{d-1} \alpha_0^j x_j = 1,$$

$$\sum_{j=0}^{d-1} \alpha_k^j x_j = 0, \quad k = 1, \dots, d-1.$$

Since the Vandermonde matrix  $A = (\alpha_k^j)$  is nonsingular,  $x$  is unique. In particular, it follows that  $x$  is real-valued; this may be seen by taking complex conjugates and observing that  $\alpha_k$  is an algebraic conjugate of  $\alpha_0$  if and only if so is  $\bar{\alpha}_k$ , hence  $\bar{x}$  solves the same system of equations.

To prove the first part of the claim, it suffices to show that there is a constant  $K_1$  such that for any  $y \in \mathbb{R}$  there is a  $v \in T(C)$  with  $|y - v| \leq K_1$ . Fix  $y$ , then we have

$$y = \sum_{j=0}^{d-1} \alpha_0^j y x_j.$$

Let  $v_j$  be an integer such that  $|v_j - y x_j| \leq 1/2$ , and let  $v = \sum_{j=0}^{d-1} \alpha_0^j v_j$ . Then

$$|y - v| = \left| \sum_{j=0}^{d-1} \alpha_0^j (y x_j - v_j) \right| \leq \frac{1}{2} \sum_{j=0}^{d-1} |\alpha_0^j| =: K_1,$$

and, for  $k = 1, \dots, d-1$ ,

$$\left| \sum_{j=0}^{d-1} \alpha_k^j v_j \right| \leq \left| \sum_{j=0}^{d-1} \alpha_k^j (y x_j - v_j) \right| + y \left| \sum_{j=0}^{d-1} \alpha_k^j x_j \right| \leq \frac{1}{2} \sum_{j=0}^{d-1} |\alpha_k^j|.$$

The claim follows if we let  $C \geq \frac{1}{2} \sum_{j=0}^{d-1} |\alpha_k^j|$ .

We now prove (2.2). It suffices to verify that there is a constant  $K_2$  such that for any  $y \in \mathbb{R}$  there are at most  $K_2$  elements of  $T(C)$  in  $[y - C, y + C]$ . Let  $a = \sum_{j=0}^{d-1} \alpha_0^j a_j$ , then the conditions that  $a \in T(C)$  and  $|y - a| \leq C$  imply that

$$A\tilde{a} - \tilde{y} \in CQ,$$

where  $\tilde{a} = (a_0, \dots, a_{d-1})^T$ ,  $\tilde{y} = (y, 0, \dots, 0)^T$ , and  $Q = [-1, 1]^d$ . In other words,  $\tilde{a} \in A^{-1}\tilde{y} + CA^{-1}Q$ . But it is clear that the number of integer lattice points contained in any translate of  $CA^{-1}Q$  is bounded by a constant.

It remains to prove (2.1). Observe first that if  $x, x' \in T(C)$ , then  $x - x' \in T(2C)$ . Thus, in view of (2.2), it is enough to prove that for any two algebraic integers  $\beta, \gamma \in \mathbb{Z}_{\mathbb{Q}}[\alpha]$  there is a  $C_1 = C_1(\beta, \gamma)$  such that if  $x, y \in T(2C)$ , then  $x\beta + y\gamma \in T(C_1)$ . By the triangle inequality, it suffices to prove this with  $y = 0$ . Let  $x \in T(C)$ , then  $x = \sum_{j=0}^{d-1} \alpha_0^j x_j$  for some  $x_j \in \mathbb{Z}$ . We also write  $\beta = \sum_{j=0}^{d-1} \alpha_0^j b_j$ , with  $b_j \in \mathbb{Z}$ . Then  $\beta y = \sum_{i,j=0}^{d-1} \alpha_0^{i+j} x_i b_j$ . We thus need to verify that

$$\left| \sum_{i,j=0}^{d-1} \alpha_k^{i+j} x_i b_j \right| \leq C_1$$

for  $k = 1, \dots, d-1$ . But the left side is equal to

$$\left| \sum_{i=0}^{d-1} \alpha_k^i x_i \right| \cdot \left| \sum_{j=0}^{d-1} \alpha_k^j b_j \right|,$$

which is bounded by  $C_1(\beta) = C \max_k \left| \sum_{j=0}^{d-1} \alpha_k^j b_j \right|$ .

**Example.** Let  $BX$  be a symmetric convex octagon whose sides have slopes  $0, -1, \infty, \sqrt{2}$ . Let also  $T(C) = \{i + j\sqrt{2} : |i - j\sqrt{2}| \leq C\}$ , and  $S = T(10) \times T(10)$ . It is easy to see that  $T(C)$  is well distributed and that (2.2) holds. Let  $x, y \in S$ , then  $x - y = (i + j\sqrt{2}, k + l\sqrt{2})$ , where  $i + j\sqrt{2}, k + l\sqrt{2} \in T(20)$ . Depending on where  $x - y$  is located, the distance from  $x$  to  $y$  will be one of the following numbers:

$$c_1 |i + j\sqrt{2}|,$$

$$c_2 |k + l\sqrt{2}|,$$

$$c_3 |(i + k) + (j + l)\sqrt{2}|,$$

$$c_4 |(i + j\sqrt{2})\sqrt{2} - (k + l\sqrt{2})| = c_4 |(2j - k) + (i - l)\sqrt{2}|.$$

Clearly, the first three belong to  $T(20 \max(c_1, c_2, c_3))$ . For the fourth one, we have

$$\begin{aligned} c_4 |(2j - k) - (i - l)\sqrt{2}| &= c_4 |(i - j\sqrt{2})\sqrt{2} - (k - l\sqrt{2})| \\ &\leq 20c_4(1 + \sqrt{2}). \end{aligned}$$

Hence all distances between points in  $S$  belong to  $T(C)$  for some  $C$  large enough, and in particular satisfy the cardinality estimate (2.2).

### §3. ADDITIVE PROPERTIES OF MULTIDIMENSIONAL SETS AND SETS WITH SPECIFIC ADDITIVE RESTRICTIONS

Let  $Y$  be a linear space over  $\mathbb{R}$  or over  $\mathbb{Q}$ . For  $A, B \subset Y$  and  $\alpha \in \mathbb{R}$  or  $\mathbb{Q}$  we denote

$$A + B = \{a + b : a \in A, b \in B\}, \quad \alpha A = \{\alpha a : a \in A\}.$$

We say that a set  $A \subset Y$  is a  $d$ -dimensional if  $A$  is contained in some  $d$ -dimensional affine subspace of  $Y$ , but in no  $d - 1$ -dimensional affine subspace of  $Y$ . We will denote the dimension of a set  $A$  by  $d_A$ .

The following result is due to Ruzsa [Ru94, Corollary 1.1].

**Lemma 3.1.** *Let  $A, B \subset \mathbb{R}^d$ ,  $|A| \leq |B|$ , and assume that  $A + B$  is  $d$ -dimensional. Then*

$$(3.1) \quad |A + B| \geq |B| + d|A| - d(d+1)/2.$$

The special case of Lemma 3.1 with  $A = B$  was proved earlier by Freiman [F73, p. 24]). In this case we also have the following corollary.

**Corollary 3.1.** *Let  $A \subset \mathbb{R}^d$ , and assume that  $|A + A| \leq K|A|$ ,  $K \leq |A|^{1/2}$ . Then the dimension of  $A$  does not exceed  $K$ .*

*Proof.* Let  $|A| = N \geq 1$ , then  $d_A \leq N - 1$ . Suppose that  $d_A > K$ . The function  $f(x) = (x+1)N - x(x+1)/2$  is increasing for  $x \leq N - 1/2$ , hence by (3.1) we have

$$KN \geq f(d_A) > f(K) = (K+1)N - \frac{K(K+1)}{2},$$

i.e.  $K(K+1) > 2N$ , which is not possible if  $K^2 \leq N$ .

We observe that Lemma 3.1, and hence also Corollary 3.1, extends to the case when  $A, B$  are subsets of a linear space  $Y$  over  $\mathbb{Q}$ . Assume that  $Y$  is  $d$ -dimensional, and take a basis  $\{e_1, \dots, e_d\}$  in  $Y$ . Consider the space  $\mathbb{R}^d$  with a basis  $\{e'_1, \dots, e'_d\}$ . We can arrange a mapping  $\Phi : Y \rightarrow Y'$  by

$$\Phi\left(\sum_{j=1}^d \alpha_j e_j\right) = \sum_{j=1}^d \alpha_j e'_j.$$

It is easy to see that  $\Phi$  is Freiman's isomorphism of any order and, in particular, of order 2: this means that for any  $y_1, y_2, z_1, z_2$  from  $Y$  the condition

$$y_1 + y_2 \neq z_1 + z_2$$

implies

$$\Phi(y_1) + \Phi(y_2) \neq \Phi(z_1) + \Phi(z_2).$$

Therefore, if  $A, B$  are finite subsets of  $Y$  and  $A' = \Phi(A), B' = \Phi(B)$ , then  $|A+B| = |A' + B'|$ , and we get the required inequality for  $|A + B|$ .

The following is a special case of [N96, Theorem 7.8].

**Lemma 3.2.** *If  $N \in \mathbb{N}$ ,  $K > 1$ ,  $A \subset Y$ , and  $B \subset Y$  satisfy*

$$(3.2) \quad \min(|A|, |B|) \geq N, \quad |A + B| \leq KN,$$

*we have*

$$|A + A| \leq K^2|A|.$$

**Corollary 3.2.** *If  $N \in \mathbb{N}$ ,  $K > 1$ , and if  $A, B \subset Y$  satisfy (3.2) for some  $K$  with  $K^2(2K^2 + 1) < N$ , then  $d_{A+B} \leq K$ . In particular,  $d_A \leq K$  and  $d_B \leq K$ .*

*Proof.* By Lemma 3.2, we have  $|A + A| \leq K^2 N$ , hence Corollary 3.1 implies that

$$d_A \leq K^2,$$

and similarly for  $B$ . Hence  $d_{A+B} \leq d_A + d_B \leq 2K^2$ . By Lemma 3.1, we have

$$\begin{aligned} KN &\geq |A + B| \geq (1 + d_{A+B})N - \frac{d_{A+B}(d_{A+B} + 1)}{2} \\ &\geq d_{A+B}N + N - K^2(2K^2 + 1) \geq d_{A+B}N, \end{aligned}$$

which proves the first inequality. To complete the proof, observe that  $d_{A+B} \geq \max(d_A, d_B)$ .

**Lemma 3.3.** *Let  $K > 0$ ,  $A$  and  $B$  be finite nonempty subsets of  $\mathbb{R}$ ,  $\alpha \in \mathbb{R} \setminus \{0\}$ . Also, suppose that the following conditions are satisfied*

$$(3.3) \quad |A - \alpha B| \leq K|B|.$$

*Then there is a set  $B' \subset B$  such that*

$$(3.4) \quad |A - \alpha B'| \leq K|B'|,$$

$$(3.5) \quad |B'| \geq |A|/K,$$

*and for any  $b_1, b_2 \in B'$  the number  $\alpha(b_1 - b_2)$  is a linear combination of differences  $a_1 - a_2$ ,  $a_1, a_2 \in A$ , with integer coefficients.*

*Proof.* Let us construct a graph  $H$  on  $B$ . We join  $b_1, b_2 \in B$  (not necessary distinct) by an edge if there are  $a_1, a_2 \in A$  such that  $a_1 - \alpha b_1 = a_2 - \alpha b_2$ . Let  $B_1, \dots, B_s$  be the components of connectedness of the graph  $H$ . Thus, for any  $j = 1, \dots, s$  and for any  $b_1, b_2 \in B_s$  there is a path connecting  $b_1$  and  $b_2$  and consisting of edges of  $H$  (a one-point path for  $b_1 = b_2$  is allowed). This implies that  $\alpha(b_1 - b_2)$  is a sum of differences  $a_1 - a_2$  for some pairs  $(a_1, a_2) \in A \times A$ . Also, denoting

$$S = A - \alpha B, \quad S_j = A - \alpha B_j,$$

we see that, by the choice of  $B_1, \dots, B_s$ , the sets  $S_j$  ( $j = 1, \dots, s$ ) are disjoint.

Since

$$|B| = \sum_{j=1}^s |B_j|, \quad |S| = \sum_{j=1}^s |S_j|,$$

there is some  $j$  such that

$$|S_j|/|B_j| \leq |S|/|B|,$$

and, by (3.3),

$$|S_j| \leq K|B_j|.$$

On the other hand,

$$|S_j| = |A - \alpha B_j| \geq |A|.$$

Hence,

$$|B_j| \geq |S_j|/K \geq |A|/K.$$

So, the set  $B' = B_j$  satisfies (3.4) and (3.5), and Lemma 3.3 follows.

**Lemma 3.4.** *Let  $K > 0$ ,  $A$  and  $B$  be finite nonempty subsets of  $\mathbb{R}$ ,  $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ . Also, suppose that the conditions*

$$(3.6) \quad |A - \alpha_1 B| \leq K|B|,$$

$$(3.7) \quad |A - \alpha_2 B| \leq K|A|,$$

*are satisfied. Then there are nonempty sets  $A' \subset A$  and  $B' \subset B$  such that*

$$(3.8) \quad |A - \alpha_1 B'| \leq K|B'|,$$

$$(3.9) \quad |A' - \alpha_2 B'| \leq K|A'|,$$

$$(3.10) \quad |A'| \geq |A|/K^2,$$

*and for any  $a'_1, a'_2 \in A'$  the difference  $a'_1 - a'_2$  is a linear combination of numbers  $\frac{\alpha_2}{\alpha_1}(a_1 - a_2)$ ,  $a_1, a_2 \in A$ , with integer coefficients.*

*Proof.* By (3.6), we can use Lemma 3.3 for  $\alpha = \alpha_1$ , and we get (3.8) and (3.5). Further, we use Lemma 3.3 again for  $B'$ ,  $A$  (thus, in the reverse order), and we get (3.9) and also

$$|A'| \geq |B'|/K.$$

Combining the last inequality with (3.5) we obtain (3.10). The proof of the lemma is complete.

Replacing (3.8) by a weaker inequality

$$|A' - \alpha_1 B'| \leq K|B'|$$

and iterating Lemma 3.4, we get the following.

**Lemma 3.5.** *Let  $K > 0$ ,  $A$  and  $B$  be finite nonempty subsets of  $\mathbb{R}$ ,  $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ . Also, suppose that the conditions (3.6) and (3.7) are satisfied. Then there are nonempty sets  $A_j \subset A$  and  $B_j \subset B$  ( $j = 0, 1, \dots$ ) such that  $A_0 = A$ ,  $B_0 = B$ ,  $A_j \subset A_{j-1}$ ,  $B_j \subset B_{j-1}$  for  $j \geq 1$ ,*

$$|A_j - \alpha_2 B_j| \leq K|A_j| \quad (j \geq 1),$$

$$|A_j| \geq |A|/K^{2j},$$

*and for any  $a_1, a_2 \in A_j$  the difference  $a_1 - a_2$  is a linear combination of numbers  $\frac{\alpha_2^j}{\alpha_1^j}(a'_1 - a'_2)$ ,  $a'_1, a'_2 \in A$ , with integer coefficients.*

Now we are in position to come to the main object of our constructions: to show that under the assumptions of Lemma 3.5, providing that the number  $\alpha_1/\alpha_2$  is transcendental, we can conclude that the dimension of the set  $A$  over  $\mathbb{Q}$  cannot be too small.

**Corollary 3.6.** *Let  $K > 0$ ,  $A$  and  $B$  be finite nonempty subsets of  $\mathbb{R}$ ,  $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$  such that  $\alpha_1/\alpha_2$  is transcendental. Also, suppose that the conditions (3.6) and (3.7) are satisfied. Then, if for some  $d \in \mathbb{N}$  the inequality*

$$(3.11) \quad |A| > K^{2d}$$

*holds, then the dimension of  $A$  over  $\mathbb{Q}$  is greater than  $d$ .*

*Proof.* By Lemma 3.5 and (3.11), we have  $|A_d| \geq 2$ . Take distinct  $a_1, a_2 \in A_d$ . Then also  $a_1, a_2 \in A_j$  for  $j = 0, 1, \dots, d$ , and, by Lemma 3.6, the difference  $a_1 - a_2$  is a linear combination of numbers  $\frac{\alpha_2^j}{\alpha_1^j}(a'_1 - a'_2)$ ,  $a'_1, a'_2 \in A$ , with integer coefficients.

Therefore, all numbers  $b_j = \frac{\alpha_2^j}{\alpha_1^j}(a_1 - a_2)$  belong to the linear span of  $a'_1 - a'_2$ ,  $a'_1, a'_2 \in A$ , over  $\mathbb{Q}$ . But, since  $\alpha_1/\alpha_2$  is transcendental, the numbers  $b_j$  ( $j = 0, \dots, d$ ) are linearly independent over  $\mathbb{Q}$ . Therefore, the dimension of the linear span of  $a'_1 - a'_2$ ,  $a'_1, a'_2 \in A$ , over  $\mathbb{Q}$  is at least  $d + 1$ , as required.

**Corollary 3.7.** *If  $A$  is a subset of  $\mathbb{R}$ ,  $2 \leq |A| < \infty$ ,  $\alpha$  is a transcendental real number, then*

$$|A - \alpha A| \gg |A| \log |A| / \log \log |A|.$$

*Proof.* Suppose that the conclusion fails, then for any  $\epsilon > 0$  we may find arbitrarily large  $N$  and  $A \subset \mathbb{R}$  with  $|A| = N$  such that

$$|A - \alpha A| \leq KN, \quad K = \epsilon \frac{\log N}{\log \log N}.$$

By Corollary 3.2, we have  $d_A \leq K$ . On the other hand, (3.6) holds with  $B = A$ ,  $\alpha_1 = \alpha$ , and, since  $A - \alpha^{-1}A = -\alpha^{-1}(A - \alpha A)$ , (3.7) holds with  $B = A$  and  $\alpha_2 = \alpha^{-1}$ . Corollary 3.7 then implies that

$$N \leq K^{2K}.$$

Taking logarithms of both sides, and assuming that  $2\epsilon < 1$ , we obtain

$$\log N \leq 2\epsilon \frac{\log N}{\log \log N} (\log(2\epsilon) + \log \log N - \log \log \log N) \leq 2\epsilon \log N,$$

which is not possible if  $N$  was chosen large enough.

**Remark.** On the other hand, if  $\alpha \in \mathbb{R}$  is an algebraic number, then one can use our construction from §2 to show that for any  $N \in \mathbb{N}$  there is a set  $A \subset \mathbb{R}$ ,  $|A| = N$ , such that

$$|A - \alpha A| \leq C|A|,$$

where  $C$  depends only on  $\alpha$ .

Finally, we state a lemma due to J. Bourgain[B99, Lemma 2.1]. For our purposes, we need a slightly more precise formulation than that given in [B99]; the required modifications are described below.

**Lemma 3.8.** *Let  $N \geq 2$ ,  $A, B$  be finite subsets of  $\mathbb{R}$  and  $G \subset A \times B$  such that*

$$(3.12) \quad |A|, |B| \leq N,$$

$$(3.13) \quad |S| \leq N \quad \text{where} \quad S = \{a + b : (a, b) \in G\},$$

$$(3.14) \quad |G| \geq \delta N^2.$$

*Then there exist  $A' \subset A$ ,  $B' \subset B$  satisfying the conditions*

$$(3.15) \quad |(A' \times B') \cap G| \gg \delta^5 N^2 (\log N)^{-C_1},$$

$$(3.16) \quad |A' - B'| \ll N^{-1} (\log N)^{C_2} \delta^{-13} |(A' \times B') \cap G|.$$

In [B], the bounds (3.15) and (3.16) involved factors of the form  $N^{\gamma+}$  and  $N^{\gamma-}$ , where  $N^{\gamma+}$  ( $N^{\gamma-}$ ) means  $\leq C(\varepsilon)N^{\gamma+\varepsilon}$  for all  $\varepsilon > 0$  and some  $C(\varepsilon) > 0$  (resp.,  $\geq c(\varepsilon)N^{\gamma-\varepsilon}$  for all  $\varepsilon > 0$ ,  $c(\varepsilon) > 0$ ). We need a slightly stronger statement, namely that the same bounds hold with the factors in question obeying the inequalities  $\ll N^\gamma (\log N)^C$  or  $\gg N^\gamma (\log N)^{-C}$ , respectively, for some appropriate choice of a constant  $C$ . A careful examination of the proof in [B99] shows that it remains valid with this new meaning of the notation  $N^{\gamma+}$  and  $N^{\gamma-}$ , and that one may in fact take  $C_1 = 5$ ,  $C_2 = 10$ . We further note that although Bourgain states his lemma for  $A, B \subset \mathbb{Z}^d$ , the same proof works for  $A, B \subset \mathbb{R}$  if the exponential sum inequality [B99, (2.7)] is replaced by

$$|G| < \int_S \chi_A * \chi_B \leq |S|^{1/2} \|\chi_A * \chi_B\|_2;$$

we then observe that

$$\begin{aligned} \|\chi_A * \chi_B\|_2^2 &= |\{(a, a', b, b') \in A \times A \times B \times B : a + b = a' + b'\}| \\ &= |\{(a, a', b, b') \in A \times A \times B \times B : a - b' = a' - b\}| = \|\chi_A * \chi_{-B}\|_2^2, \end{aligned}$$

and proceed further as in [B99]. A similar modification should be made in [B99, (2.36)].

#### §4. PROOF OF THEOREM 4

In this section we prove Theorem 4; note that this also proves the implication (ii)  $\Rightarrow$  (i) of Theorem 3.

Suppose that  $BX$  is a polygon with finitely many sides for which the conclusion of the theorem fails, i.e. that there is a well distributed set  $S$  such that for any  $\epsilon > 0$  there is an increasing sequence of positive integers  $N_1, N_2, \dots \rightarrow \infty$  with

$$(4.1) \quad |\Delta_{X, N_j}(S)| < \epsilon N_j \psi(N_j),$$

where

$$\psi(N) = \log N / \log \log N.$$

Without loss of generality we may assume that  $\partial BX$  contains a vertical line segment and a horizontal line segment, and that  $c_1 Bl_2^2 \subset BX \subset Bl_2^2$ . Let also  $c_2 \in (0, 1/10)$  be a small constant such that all sides of  $BX$  have length at least  $8c_2$ .

Let  $M$  be a sufficiently large number which may depend on  $\epsilon$ ; all other constants in the proof will be independent of  $\epsilon$ . Let  $T = N_{j_0}$  for some  $j_0$  large enough so that  $T > M$ , and let  $N = c_2 T$ . Suppose that one of the two vertical sides of  $BX$  is the line segment  $\{(x_1, x_2) : x_1 = v_1, |x_2 - v_2| \leq r\}$ , where  $v_1 > 0$ . Let also  $Q = \text{Int}(N \cdot BX)$ ,  $v = (v_1, v_2)$ , and

$$A = \{x_1 : (x_1, x_2) \in S \cap Q \text{ for some } x_2\},$$

$$Q' = Q + (T - 2N)v.$$

Observe that both  $Q$  and  $Q'$  have Euclidean diameter  $\leq 2N$ , and that

$$Q' \subset \{(x_1, x_2) : (T - 3N)v_1 < x_1 < (T - N)v_1\},$$

so that

$$\|x - x'\|_X \geq (1 - 4c_2)T > T/2, \quad x \in Q, x' \in Q'.$$

By our choice of  $c_2$  we have  $c_2 \leq r/4$ , so that

$$T/2 \cdot r \geq 2N.$$

Hence all  $X$ -distances between points in  $Q$  and  $Q'$  are measured using the vertical segments of  $\partial BX$ , i.e.

$$\|x - x'\|_X = |x_1 - x'_1|/v_1, \quad x = (x_1, x_2) \in Q, x' = (x'_1, x'_2) \in Q'.$$

Next, we claim that

$$(4.2) \quad |\{\|x - x'\|_X : x \in S \cap Q, x' \in S \cap Q'\}| < K_0 \epsilon N \psi(N),$$

where  $K_0$  is a constant depending only on  $c_2$ . Indeed, we have

$$\{\|x - x'\|_X : x \in Q, x' \in Q'\} \subset [0, T],$$

hence the failure of (4.2) would imply that

$$|\Delta_{X,T}(S)| \geq K_0 \epsilon N \psi(N) \geq \epsilon T \psi(T),$$

if  $K_0$  is large enough (at the last step we used that  $\psi(N) \gg \psi(c_2^{-1}N) = \psi(T)$ ). But this contradicts (4.1).

It follows that if we define

$$A' = \{x'_1 : (x'_1, x'_2) \in S \cap Q' \text{ for some } x'_2\},$$

then we can estimate the cardinality of the difference set  $A - A'$  using (4.2):

$$(4.3) \quad |A - A'| < K_0 \epsilon N \psi(N).$$

On the other hand, since  $S$  is well distributed, we must have

$$(4.4) \quad |A|, |A'| \gg N.$$

Hence by Corollary 3.2 we have

$$(4.5) \quad d_A \ll \epsilon \psi(N).$$

We may now repeat the same argument with the vertical side of  $\partial BX$  replaced by its other sides. In particular, using the horizontal segment in  $\partial BX$  instead, we obtain the following. Let

$$B = \{x_2 : (x_1, x_2) \in S \cap Q \text{ for some } x_1\},$$

then there is a set  $B' \subset \mathbb{R}$  such that

$$(4.6) \quad |B|, |B'| \gg N,$$

$$(4.7) \quad |B - B'| < K_0 \epsilon N \psi(N),$$

$$(4.8) \quad d_B \ll \epsilon \psi(N).$$

Furthermore, assume that  $\partial BX$  contains a segment of a line  $x_1 + \alpha x_2 = \beta$ , then

$$(4.9) \quad |\{x_1 + \alpha x_2 : (x_1, x_2) \in S \cap Q\}| \leq K_0 \epsilon N \psi(N);$$

this estimate is an easier analogue of (4.3) obtained by counting distances between points in  $Q$  and just one point in the appropriate analogue of  $Q'$ .

Suppose that  $\partial BX$  contains segments of lines  $x_1 + \alpha_1 x_2 = C_1$ ,  $x_2 + \alpha_2 x_1 = C_2$  (i.e. with slopes  $-1/\alpha_1$ ,  $-1/\alpha_2$ ), where  $\alpha_1, \alpha_2$  are neither 0 nor  $\infty$ , and that the ratio  $\alpha_1/\alpha_2$  is transcendental. Let  $G = (A \times B) \cap S$ , then  $|G| \geq c_4 N^2$  since  $S$  is well distributed. By (4.4), (4.6), and (4.9) with  $\alpha = \alpha_1$ , the assumptions of Lemma 3.8 are satisfied with  $N$  replaced by  $K_0 \epsilon N \psi(N)$  and  $\delta = c_4 (K_0 \epsilon \psi(N))^{-2}$ . We conclude that there are subsets  $A_1 \subset A$  and  $B_1 \subset B$  such that

$$(4.10) \quad |(A_1 \times B_1) \cap G| \gg N^2 \epsilon^c (\log N)^{-c},$$

$$(4.11) \quad |A_1 - \alpha_1 B_1| \ll N^{-1} \epsilon^{-c} (\log N)^c |(A_1 \times B_1) \cap G|.$$

Here and below,  $c$  denotes a constant which may change from line to line but is always independent of  $N$ . We also simplified the right sides of (4.10) and (4.11) by noting that  $\psi(N) \leq \log N$ .

Similarly, applying Lemma 3.8 with  $G$  replaced by  $(A_1 \times B_1) \cap G$  and  $\alpha_1$  replaced by  $\alpha_2$ , we find subsets  $A_2 \subset A_1$  and  $B_2 \subset B_1$  such that

$$(4.12) \quad |(A_2 \times B_2) \cap G| \gg N^2 \epsilon^c (\log N)^{-c},$$

$$(4.13) \quad |A_2 - \alpha_2 B_2| \ll N^{-1} \epsilon^{-c} (\log N)^c |(A_2 \times B_2) \cap G|.$$

Clearly, (4.11) also holds with  $A_1, B_1$  replaced by  $A_2, B_2$ .

Thus  $A_2, B_2$  satisfy the assumptions (3.14), (3.15) of Corollary 3.7, with  $K = \epsilon^{-c} (\log N)^c$ . By (4.4), (4.5) and Corollary 3.7, we must have for some constants  $c, K_2$ ,

$$cN \leq |A_2| < (\epsilon^{-1} \log N)^{K_2 \epsilon \log N / \log \log N},$$

hence

$$\log c + \log N \leq \frac{K_2 \epsilon \log N}{\log \log N} (\log \log N - \log \epsilon) \leq 2K_2 \epsilon \log N,$$

a contradiction if  $\epsilon$  was chosen small enough. This proves that if (0.5) fails, then the ratio between any two slopes, other than 0 or  $\infty$ , of sides of  $BX$  is algebraic.

To conclude the proof of the theorem, we first observe that if  $BX$  is a rectangle, then there is nothing to prove. If  $BX$  is a hexagon with slopes  $0, \infty, \alpha$ , we may always find a coordinate system as in Theorem 3 (i); namely, if we let

$$(4.14) \quad x'_1 = x_1, \quad x'_2 = \alpha x_2,$$

then the slopes 0 and  $\infty$  remain unchanged, and lines  $\alpha x_1 - x_2 = C$  with slope  $\alpha$  are mapped to lines  $x'_1 - x'_2 = C/\alpha$  with slope 1. Finally, suppose that  $BX$  is a polygon with slopes  $0, \infty, \alpha_1, \alpha_2, \dots, \alpha_l$ , and apply the linear transformation (4.14) with  $\alpha = \alpha_1$ . Then the sides of  $\partial BX$  with slope  $\alpha_1$  is mapped to line segments with slope 1; moreover, since the ratios  $\alpha_j/\alpha_1$ ,  $j = 2, 3, \dots, l$ , remain unchanged in the new coordinates, and since we have proved that these ratios are algebraic, all remaining sides of  $\partial BX$  are mapped to line segments with algebraic slopes.

**Acknowledgements.** This work was completed while the first author was a PIMS Distinguished Chair at the University of British Columbia, and was partially supported by NSERC grant 22R80520. We are indebted to Ben Green for pointing out to us the reference [Ru].

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